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基本超几何级数的求和

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摘　要　本文运用基本超几何级数求和的一个简单算法,求得一些基本超几何级数的求和公式.

关键词　基本超几何级数;求和;Γ函数
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0　引　言

设 a,q 是复数,且 $0<q<1$,定义:

$$(a)_n=(a;q)_n=\prod_{i=1}^{n}(1-aq^{i-1}),(a)_0=1,(a)_\infty$$
$$=\prod_{i=1}^{\infty}(1-a^{i-1}),(a_1,a_2,\cdots,a_m;q)_\beta=\prod_{i=1}^{m}(a_i;q)_\beta,$$
$${}_r\Phi_s=\left[\begin{matrix}a_1,a_2,\cdots,a_r\\b_1,b_2,\cdots,b_s\end{matrix};q,z\right]=\sum_{n=0}^{\infty}\prod_{i=1}^{r}(a_i)_n\Big/\prod_{i=1}^{s}(b_i)_n\cdot((-1)^nq^{\binom{n}{2}})^{1+s-r}\cdot\frac{z^n}{(q)_n}$$

此处当 $r=1+s$, $|z|<1$;当 $r\leqslant s$, $|z|<+\infty$

$${}_{r+1}W_r(a;a_1,\cdots,a_{r-2};q,z)={}_{r+1}\Phi_r\left[\begin{matrix}a,\sqrt{a}q,-\sqrt{a}q,a_1,\cdots,a_{r-2}\\\sqrt{a},-\sqrt{a},aq/a_1,\cdots,aq/a_{r-2}\end{matrix};q,z\right].$$

定义:

$$\tau_q(x)=\begin{cases}\dfrac{(q)_\infty}{(q^x)_\infty}(1-q)^{1-x} & |q|<1\\[2ex]\dfrac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x}q^{\binom{x}{2}} & q>1\end{cases}$$

$$\Psi_q(x)=\frac{d}{dx}\ln|\tau_q(x)|.$$

当 $0<q<1$ 时,易推出 $\Psi_q(x)=-\ln(1-q)+\sum\limits_{n=0}^{\infty}\dfrac{q^{x+n}/\ln^q}{1-q^{x+n}}$.文献(1)中,运用了一个简单算法,得到了一些基本超几何级数的求和公式,这个简单算法为

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$$
{}_{r+1}\Phi_{s+1}\left[\begin{matrix} a_1q, a_2q, \cdots, a_rq, q \\ b_1q, b_2q, \cdots, b_sq, q^2 \end{matrix}; q, q^{1+s-r}z\right]
$$

$$
= \frac{(-1)^{1+s-r}(1-q)}{z}\cdot\frac{\prod_{i=1}^{s}(1-b_i)}{\prod_{i=1}^{r}(1-a_i)}\left\{{}_r\Phi_s\left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix}; q, z\right]-1\right\}.
$$

本文继续使用这个算法,来对一些基本超几何级数进行求和.

1 几个求和公式

令 $\widetilde{\Psi}_q(x)=(q-1)\sum_{n=0}^{\infty}\frac{xq^n}{1-xq^n}(0<q<1)$,则有

$$
\widetilde{\Psi}_q(x)=(q-1)\frac{\ln(1-q)}{\ln q}+\frac{(q-1)}{\ln q}\Psi_q(x).
$$

定理 1 $\lim\limits_{N\to+\infty}\left(\widetilde{\Psi}_q\left(\frac{x}{q^N}\right)-\widetilde{\Psi}_q\left(\frac{y}{q^N}\right)\right)=\widetilde{\Psi}_q(x)-\widetilde{\Psi}_q(y)$

证明 依据 $\widetilde{\Psi}_q(x)$ 的表示,易推出

$$
\widetilde{\Psi}_q(x)-\widetilde{\Psi}_q(y)-\sum_{n=0}^{\infty}\frac{q^n(y-x)(1-q)}{(1-xq^n)(1-yq^n)}
$$

$$
\widetilde{\Psi}_q(x)-\widetilde{\Psi}_q(xq^N)=(q-1)\sum_{n=0}^{N-1}\frac{xq^n}{1-xq^n}
$$

则 $\widetilde{\Psi}_q\left(\frac{x}{q^N}\right)-\widetilde{\Psi}_q(x)\left(\frac{y}{q^N}\right)=\left(\widetilde{\Psi}_q\left(\frac{x}{q^N}\right)-\widetilde{\Psi}_q(x)\right)-\left(\widetilde{\Psi}_q\left(\frac{y}{q^N}\right)-\widetilde{\Psi}_q(y)\right)+\widetilde{\Psi}_q(x)-\widetilde{\Psi}_q(y)$

$$
=(q-1)\sum_{n=0}^{N-1}\left[\frac{\frac{x}{q}\cdot\frac{1}{q^{N-n-1}}}{1-\frac{x}{q}\cdot\frac{1}{q^{N-n-1}}}-\frac{\frac{y}{q}\cdot\frac{1}{q^{N-n-1}}}{1-\frac{y}{q}\cdot\frac{1}{q^{N-n+1}}}\right]+\widetilde{\Psi}_q(x)-\widetilde{\Psi}_q(y)
$$

$$
=(q-1)\sum_{n=0}^{N-1}\left[\frac{\frac{x}{q}\cdot\frac{1}{q^n}}{1-\frac{x}{q}\cdot\frac{1}{q^n}}-\frac{\frac{y}{q}\cdot\frac{1}{q^n}}{1-\frac{y}{q}\cdot\frac{1}{q^n}}\right]+\widetilde{\Psi}_q(x)-\widetilde{\Psi}_q(y)
$$

$$
=\sum_{n=0}^{N-1}\frac{q^n(1-q)(q/x-q/y)}{(1-q^{n+1}/x)(1-q^{n+1}/y)}+\widetilde{\Psi}_q(x)-\widetilde{\Psi}_q(y)
$$

$$
=\widetilde{\Psi}_q\left(\frac{q}{x}\right)-\widetilde{\Psi}_q\left(\frac{q}{y}\right)+\widetilde{\Psi}_q(x)-\widetilde{\Psi}_q(y).
$$

定理 2 ${}_3\Phi_2\left[\begin{matrix} aq, q, q \\ bq, q^2 \end{matrix}; q, \frac{b}{a}\right]=\sum_{n=0}^{\infty}\frac{a(1-b)(1-q)q^n}{(1-bq^n)(a-bq^n)}$.

证明 由文献(1)知

$$
{}_4\Phi_3\left[\begin{matrix} aq, cq, q, q \\ bq, acq^2/b, q^2 \end{matrix}; q, q\right]=\frac{1}{q}\frac{(1-b)(1-qac/b)}{(1-a)(1-c)}\cdot\left\{\widetilde{\Psi}_q\left(\frac{q}{b}\right)-\widetilde{\Psi}_q\left(\frac{cq}{b}\right)-\widetilde{\Psi}_q\left(\frac{aq}{b}\right)+\right.
$$

$$
\left.\widetilde{\Psi}_q\left(\frac{acq}{b}\right)-\frac{(1-q)(a, c, q, acq^2/b^2; q)_\infty}{(b/q, aq/b, cq/b, acq/b; q)_\infty}\cdot{}_3\Phi x\left[\begin{matrix} aq/b, cq/b, q/b \\ q^2/b, acq^2/b^2 \end{matrix}; q, q\right]\right\}
$$

(其中 $(a_1,\cdots,a_m;q)_\beta=\prod_{i=1}^{m}(a_i,q)_\beta$)

令 $c=q^{-N}$ 且 $N\to+\infty$,由定理1,有

$$ {}_3\Phi_2\left[\begin{matrix}aq,q,q\\ bq,q^2\end{matrix};q\right]=\frac{(1-1/b)}{(1-1/a)}\left(\widetilde{\Psi}_q(b)-\widetilde{\Psi}_q\left(\frac{b}{a}\right)\right) $$

而

$$ \widetilde{\Psi}_q(b)-\widetilde{\Psi}_q\left(\frac{b}{a}\right)=\sum_{n=0}^{\infty}\frac{q^n\left(\frac{b}{a}-b\right)(1-q)}{(1-bq^n)\left(1-\frac{b}{a}q^n\right)} $$

$$ =\sum_{n=0}^{\infty}\frac{baq^n\left(\frac{1}{a}-1\right)(1-q)}{(1-bq^n)(a-bq^n)} $$

故

$$ {}_3\Phi_2\left[\begin{matrix}aq,q,q\\ bq,q^2\end{matrix};q,\frac{b}{a}\right]=\sum_{n=0}^{\infty}\frac{a(1-b)(1-q)q^n}{(1-bq^n)(a-bq^n)}. $$

定理3 $$ {}_5\Phi_4\left[\begin{matrix}aq,-\sqrt{a}q^2,bq,q,q\\ -\sqrt{a}q,aq^2/b,aq^2,q^2\end{matrix};q,\frac{q\sqrt{a}}{b}\right]=\frac{b(aq-b)(1-aq)(1-q)}{(1-b)(1+\sqrt{a}q)} $$
$$ \left\{\sum_{n=0}^{\infty}\left[\frac{1/b}{(1-\sqrt{a}q^{n+1})(1-aq^{n+1})}-\frac{1}{(b-\sqrt{a}q^{n+1})(b-aq^{n+1})}\right]\right\}. $$

证明 由文献(2),可得:

$$ {}_9\Phi_8\left[\begin{matrix}\sqrt{a}q^2,-\sqrt{a}q^2,bq,cq,dq,a^2q^2/bcd,aq,q,q\\ \sqrt{a}q,-\sqrt{a}q,aq^2/b,aq^2/c,aq^2/d,bcdq/a,q^2,aq^2\end{matrix};q,q\right] $$
$$ =\frac{1}{q}\frac{(1-aq/b)(1-aq/c)(1-aq/d)(1-aq)(1-bcd/a)}{(1-q^2a)(1-b)(1-c)(1-d)(1-qa^2/bcd)}\times $$
$$ \left[\widetilde{\Psi}_q\left(\frac{aq}{d}\right)-\widetilde{\Psi}_q(aq)+\widetilde{\Psi}_q\left(\frac{aq}{c}\right)-\widetilde{\Psi}_q\left(\frac{aq}{cd}\right)+\widetilde{\Psi}_q\left(\frac{b}{a}\right)-\widetilde{\Psi}_q\left(\frac{bd}{a}\right)+\widetilde{\Psi}_q\left(\frac{bcd}{a}\right)-\widetilde{\Psi}_q\left(\frac{bc}{a}\right)-\right. $$
$$ \frac{(1-q)(c,d,q,a^2q/bcd,bq/c,bq/d,bq,b^2cd/a^2;q)_\infty}{(a/b,aq/c,aq/d,bcd/a,bc/a,bd/a,aq/cd,b^2q/a;q)_\infty}\times $$
$$ \left.{}_8W_7\left(\frac{b^2}{a};b,\frac{bc}{a},\frac{bd}{a},\frac{b}{a},\frac{aq}{cd};q,q\right)\right] $$

令 $d=q^{-N}$ 且 $N\to+\infty$,再由定理1

可得:

$$ {}_7\Phi_6\left[\begin{matrix}aq,\sqrt{a}q^2,-\sqrt{a}q^2,bq,cq,q,q\\ \sqrt{a}q,-\sqrt{a}q,aq^2/b,aq^2/c,aq^2,q^2\end{matrix};q,\frac{qa}{bc}\right] $$
$$ =\frac{bc}{qa}\frac{(1-aq/b)(1-aq/c)(1-aq)}{(1-aq^2)(1-b)(1-c)}\left[\widetilde{\Psi}_q\left(\frac{aq}{c}q\right)-\widetilde{\Psi}_q(aq)+\widetilde{\Psi}_q\left(\frac{aq}{b}\right)-\widetilde{\Psi}_q\left(\frac{aq}{bc}\right)\right]. $$

又设 $c=\sqrt{a}$,则可得:

$$ {}_5\Phi_4\left[\begin{matrix}aq,-\sqrt{a}q^2,bq,q,q\\ -\sqrt{a}q,aq^2/b,aq^2,q^2\end{matrix};q,q\sqrt{a}/b\right] $$
$$ =\frac{b}{\sqrt{a}q}\frac{(1-aq/b)(1-aq)(1+\sqrt{a})}{(1-b)(1-a)(1+\sqrt{a}q)}\left[\widetilde{\Psi}_q(\sqrt{a}q)-\widetilde{\Psi}_q(aq)+\widetilde{\Psi}_q\left(\frac{aq}{b}\right)-\widetilde{\Psi}_q\left(\frac{\sqrt{a}q}{b}\right)\right] $$

而
$$\widetilde{\Psi_q}(\sqrt{a}q)-\widetilde{\Psi_q}(aq)=\sum_{n=0}^{\infty}\frac{q^n(aq-\sqrt{a}q)(1-q)}{(1-\sqrt{a}q^{n+1})(1-aq^{n+1})},$$
$$\widetilde{\Psi_q}\left(\frac{aq}{b}\right)-\widetilde{\Psi_q}\left(\frac{\sqrt{a}q}{b}\right)=\sum_{n=0}^{\infty}\frac{bq^n(1-q)(\sqrt{a}q-aq)}{(b-\sqrt{a}q^{n+1})(b-aq^{n+1})},$$
代人后，化简即得：
$${}_5\Phi_4\left[\begin{matrix}aq,-\sqrt{a}q^2,bq,q,q\\-\sqrt{a}q,aq^2/b,aq^2,q^2\end{matrix};q,\frac{\sqrt{a}q}{b}\right]=\frac{b(aq-b)(1-aq)(1-q)}{(1-b)(1+\sqrt{a}q)}$$
$$\left\{\sum_{n=0}^{\infty}\left[\frac{1/b}{(1-\sqrt{a}q^{n+1})(1-aq^{n+1})}-\frac{1}{(b-\sqrt{a}q^{n+1})(b-aq^{n+1})}\right]\right\}.$$

推论 $${}_4\Phi_4\left[\begin{matrix}aq,-\sqrt{a}q^2,q,q\\-\sqrt{a}q,aq^2,q^2,0\end{matrix};q,\sqrt{a}q^2\right]$$
$$=\frac{q^n(1-q)(1-aq)}{(1+\sqrt{a}q)}\cdot\sum_{n=0}^{\infty}\frac{1}{(1-aq^{n+1})(1-\sqrt{a}q^{n+1})}.$$

证明 以 $b=\sqrt{a}$ 代人定理3后，即可得出.

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The Summations for Basic Hypergeometric Series

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Abstract By using a simple algorithm for the summation of basic hypergeometric series, summation formulas for some basic hypergeometric series are obtained.

Key words basic hypergeometric series; summation; gamma function